Weighted Coordinate Transformations And Their Applications In SPCS And Boundary Surveying

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Weighted Coordinate Transformations and their applications

ABSTRACT

Coordinate transformation is the process of determining the relationship between two sets of coordinates. It answers the question: "I have a point with coordinates from one source, what are the coordinate values for this point on another source of information".

Weighted Coordinate Transformations and their applications

ABSTRACT (CONTINUED)

The input to the process are coordinates of common points from both coordinate systems (with or without weights) and the output is a set of parameters that define the shifts, rotations and scale difference between these systems. An important objective of this procedure is to keep it as simple as possible (essentially filling out a spreadsheet) so that it can be easily adopted by the Professional Land Surveyor.

Weighted Coordinate Transformations and their applications

ABSTRACT (CONTINUED)

Weighted coordinate transformation has many application in surveying practice. The first is to compute NAD83 coordinate values for points with NAD27 coordinates. The second is to combine maps from different sources when building a GIS. The third application is to sort out evidence in boundary surveys.

Seminar OUTLINE

- Coordinate systems
- Weights
- The mathematics of coordinate transformation
- Application I: NAD27 to NAD83 and back
- Application II: Evaluation of boundary evidence
- Application III: GIS

Weighted Coordinate Transformations and their applications

Coordinate Transformation answers the question:

"I know the coordinates of a point on one data source, what are the coordinate values for this point on another data source".

Points have different accuracies, therefore, they must be weighted accordingly

Weighted Coordinate Transformations and their applications

Objectives:

- Least Squares solution
- Should be easily adopted by PLS or GIS technical Staff

Weighted Coordinate Transformations and their applications

Applications

- Data Conversion
  - Map to GIS
  - Orthophoto to GIS
- NAD 27 <=> NAD 83
- Sort out evidence in boundary survey
- Apply Scale Factor for plan stakeout

Weighted Coordinate Transformations and their applications

Weights of Measurements

Some measurements are more accurate than others due to:

- Quality of equipment
- Better techniques
- Better field conditions
- better observer (experience and capability)

Weighted Coordinate Transformations and their applications

Elements that contribute to differences in the accuracies of a computed product are:

- Office computation procedure
  - (i.e. traverse computation with least squares vs. the Compass Rule)
- Quality of external information
  - (i.e. accuracy of a bench mark used for elevation)

Weights of Measurements

Measurement	"Strong"	"Weak"
Weight	High	Low
Precision	High	Low
Influence	High	Low
Standard Error	Low	High

Methods for Determining Weights

- Reciprocal of the Variance or $1/\sigma^2$
- Number of Setups
- Experience
- Computing Weighted Statistics

Computing Weighted Statistics

A. The Weighted Mean

$$\bar{x} = \frac{\sum (w_i \cdot x_i)}{\sum w_i}$$

B. The Standard Deviation of the Weighted Mean

$$\bar{s} = \sqrt{\frac{\sum (w_i \cdot v_i^2)}{(n-1) \cdot \sum w_i}}$$

Computing Weighted Statistics

C. The Standard Error for Each Observation

$$s_i = \sqrt{\frac{\sum (w_i \cdot v_i^2)}{(n-1) \cdot w_i}}$$

or $s_i = \dfrac{s_o}{\sqrt{w_i}}$ Where $s_o = \sqrt{\dfrac{\sum (w_i \cdot v_i^2)}{(n-1)}}$

Weights for coordinate transformation

Application	Source for weights
NAD27 to NAD83	Accuracy and Standards
Map to GIS	Map (Digitization) Accuracies
Boundary	Hierarchy of Evidence and Experience

Weights for coordinate transformation

Hierarchy of Evidence:

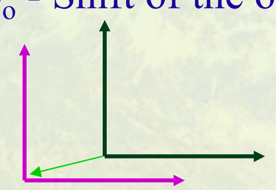
- Call for monuments
  - Natural
  - Artificial
  - Record
- Course
- Distance
- Area
- Coordinates

2-D Coordinate transformation

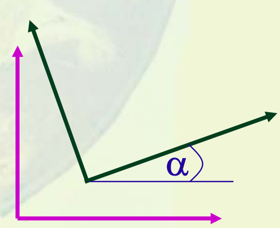
Coordinate transformation is the process of modifying one set of coordinates to make them fit another. In general there are six parameters that can change from one coordinate system to another.

2-D Coordinate transformation Parameters

1. a_o - Shift of the origin in X direction.
2. b_o - Shift of the origin in Y direction.

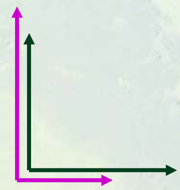


3. α - Rotation of the axes of one coordinate system with respect to the other.

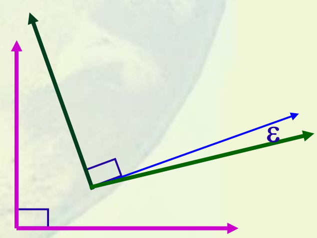


2-D Coordinate transformation Parameters

4. Scale change in X axis.
5. Scale change in Y axis.

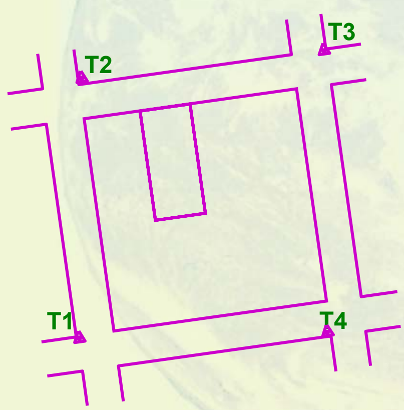


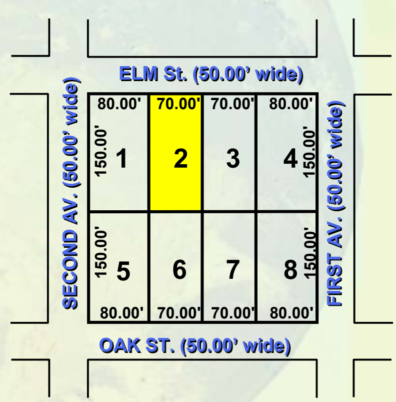
6. ε - Non-orthogonality angle between the axes of the two coordinate systems.



The Process of Transformation

Shift





2-D Coordinate transformation

- The most simple (and restrictive) is to allow only a shift of the origin or a rotation.
- The least constraining is to allow changes in all six parameters

Other names for the transformation procedures are:

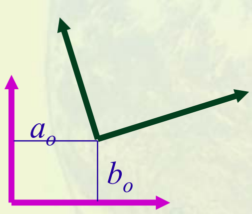
Parameters	Name	Typical Application
3	Rigid Body	Two well defined systems of exactly the same scale.
4	Similarity Conformal Helmert Isogonal	Two well defined systems that may have different scales.

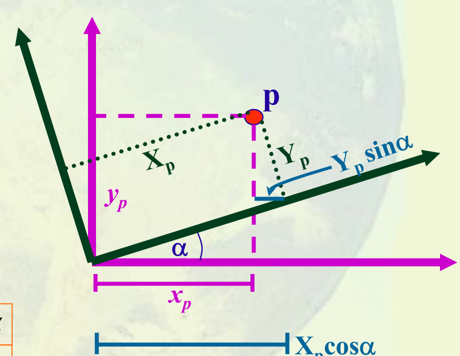
Other names for the transformation procedures are:

Parameters	Name	Typical Application
5	Orthogonal	One system may have different scales in X and Y
6	Affine	One system has unknown characteristics

Coordinate Transformation Formulas

- 3 Parameter Transformation (Rigid Body)





$$
\begin{aligned}
X &= a_o + x \cdot \cos\alpha - y \cdot \sin\alpha \\
Y &= b_o + x \cdot \sin\alpha + y \cdot \cos\alpha
\end{aligned}
$$

Coordinate Transformation Formulas

- 4 Parameter Transformation (Similarity, Conformal, Helmert, Isogonal, Euclidean)

$$X = a_o + s \cdot x \cdot \cos\alpha - s \cdot y \cdot \sin\alpha$$
$$Y = b_o + s \cdot x \cdot \sin\alpha + s \cdot y \cdot \cos\alpha$$

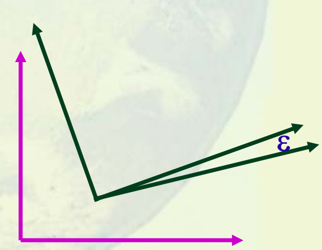
- 5 Parameter Transformation (Orthogonal)

$$X = a_o + s_x \cdot x \cdot \cos\alpha - s_y \cdot y \cdot \sin\alpha$$
$$Y = b_o + s_x \cdot x \cdot \sin\alpha + s_y \cdot y \cdot \cos\alpha$$

Coordinate Transformation Formulas

- 6 Parameter Transformation (Affine)

$$X = a_o + s_x \cdot x \cdot \cos\alpha - s_y \cdot y \cdot (\sin\alpha + \sin\varepsilon \cdot \cos\alpha)$$
$$Y = b_o + s_x \cdot x \cdot \sin\alpha + s_y \cdot y \cdot (\cos\alpha - \sin\varepsilon \cdot \sin\alpha)$$



Plane Similarity Transformation

Math Model

$$X = a_o + a_1 x - b_1 y$$
$$Y = b_o + b_1 x + a_1 y$$

Solution

Step 1

$\bar{x} = x_2 - x_1$ $\bar{y} = y_2 - y_1$ $\bar{X} = X_2 - X_1$ $\bar{Y} = Y_2 - Y_1$

Step 2

$$a_1 = \frac{\bar{X}\cdot\bar{x} + \bar{Y}\cdot\bar{y}}{\bar{x}^2 + \bar{y}^2} \qquad b_1 = \frac{\bar{Y}\cdot\bar{x} - \bar{X}\cdot\bar{y}}{\bar{x}^2 + \bar{y}^2}$$

Plane Similarity Transformation

Step 3

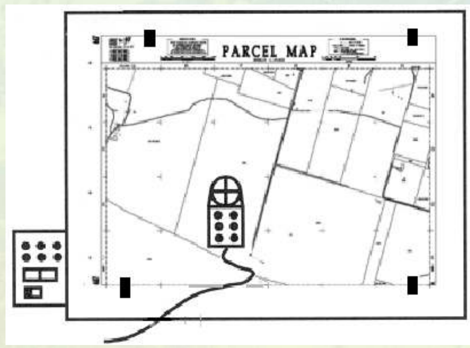
$$a_o = X_1 - a_1 x_1 + b_1 y_1 \qquad b_o = Y_1 - b_1 x_1 - a_1 y_1$$

Computing Scale and Rotation

$$\text{Scale} = \sqrt{a_1^2 + b_1^2}$$

$$\text{Rot} = \tan^{-1}\left(\frac{b_1}{a_1}\right)$$

Computation Example



Converting digitized points (x_1,y_1), (x_2,y_2)
to
State Plane Coordinates (X_1,Y_1), (X_2,Y_2)

Coordinate Transformations (similarity 4 parameters)

Point	x	y	X	Y
1	100.69	166.39	670305	224173
2	34.90	56.07	670764	223898

$\bar{x}$	$\bar{y}$	$\bar{X}$	$\bar{Y}$
-65.79	-110.32	459	-275

$\bar{x}\,\bar{X}$	$\bar{y}\,\bar{Y}$	$\bar{x}\,\bar{Y}$	$\bar{y}\,\bar{X}$	$\bar{x}_i^2 + \bar{y}_i^2$
-30197.61	30338.00	18092.25	-50636.88	16498.83

Coordinate Transformations (similarity 4 parameters)

$a_1 = 0.00851$
$b_1 = 4.1657$
$a_0 = 670997$
$b_0 = 223752$

$$a_1 = \frac{\overline{X}\cdot\overline{x} + \overline{Y}\cdot\overline{y}}{\overline{x}^2 + \overline{y}^2}$$

$$b_1 = \frac{\overline{Y}\cdot\overline{x} - \overline{X}\cdot\overline{y}}{\overline{x}^2 + \overline{y}^2}$$

$$a_o = X_1 - a_1x_1 + b_1y_1$$

$$b_o = Y_1 - b_1x_1 - a_1y_1$$

Example of using these parameters:

$X = a_0 + a_1x - b_1y$	670305	670764
$Y = b_0 + b_1x + a_1y$	224173	223898

4-Parameter **Transformation** with Over Determination

Math Model

$$X' = a_o + a_1x - b_1y$$

$$Y' = b_o + b_1x + a_1y$$

<u>Solution</u>

Step 1: Computing averages

$$x_s = \frac{\sum_{i=1}^{n} x_i}{n} \qquad y_s = \frac{\sum_{i=1}^{n} y_i}{n} \qquad X_s = \frac{\sum_{i=1}^{n} X_i}{n} \qquad Y_s = \frac{\sum_{i=1}^{n} Y_i}{n}$$

4-Parameter **Transformation** with Over Determination

Step 2: **Computing Residuals**

$$\overline{x_i} = x_i - x_s$$

$$\overline{y_i} = y_i - y_s$$

$$\overline{X_i} = X_i - X_s$$

$$\overline{Y_i} = Y_i - Y_s$$

4-Parameter **Transformation** with Over Determination

Step 3 **Computing *a₁* and *b₁***

$$I = \sum_{i=1}^{n} (\overline{X}_i \cdot \overline{x}_i)$$

$$II = \sum_{i=1}^{n} (\overline{Y}_i \cdot \overline{y}_i)$$

$$III = \sum_{i=1}^{n} (\overline{Y}_i \cdot \overline{x}_i)$$

$$IV = \sum_{i=1}^{n} (\overline{X}_i \cdot \overline{y}_i)$$

$$V = \sum_{i=1}^{n} (\overline{x}_i^2 + \overline{y}_i^2)$$

$$a_1 = \frac{I + II}{V}$$

$$b_1 = \frac{III - IV}{V}$$

4-Parameter **Transformation** with Over Determination

Step 4 **Computing a_o and b_o**

$$a_o = X_s - a_1 x_s + b_1 y_s$$

$$b_o = Y_s - b_1 x_s - a_1 y_s$$

4-Parameter **Transformation** with Over Determination

Accuracy Assessment

$$V_{x_i} = X_i - X_i^{'}$$

$$V_{y_i} = Y_i - Y_i^{'}$$

$$VI = \sum_{i=1}^{n} V_{x_i}^2$$

$$VII = \sum_{i=1}^{n} V_{y_i}^2$$

$$m_0 = m_x = m_y = \sqrt{\frac{VI + VII}{2n - 4}}$$

$$m_p = m_0 \cdot \sqrt{2}$$

Same Problem with 3 points

Point	x	y	X	Y
1	100.69	166.39	670305	224173
2	34.90	56.07	670764	223898
3	171.36	58.87	670741	224444
Σ=	306.95	281.33	2011810	672515

x_s	y_s	X_s	Y_s
102.32	93.777	670603	224172

Same Problem with 3 points

i	$\overline{x}_i$	$\overline{y}_i$	$\overline{X}_i$	$\overline{Y}_i$
1	-1.63	72.61	-298.33	1.33
2	-67.42	-37.71	160.67	-273.67
3	69.04	-34.91	137.67	272.33
Σ=	0	0	0	0

Same Problem with 3 points

i	$\bar{x}_i \cdot \bar{X}_i$	$\bar{y}_i \cdot \bar{Y}_i$	$\bar{x}_i \cdot \bar{Y}_i$	$\bar{y}_i \cdot \bar{X}_i$	$\bar{x}_i^2 + \bar{y}_i^2$
1	485.29	96.82	-2.17	-21662.98	5275.34
2	-10831.61	10319.06	18449.69	-6058.20	5966.80
3	9504.97	-9506.25	18802.80	-4805.48	5985.46
Σ=	-841.36	909.63	37250.33	-32526.67	17227.60
	I	II	III	IV	V

Same Problem with 3 points

Solution: $a_0 = 670983$
$a_1 = 0.00396$
$b_0 = 223757$
$b_1 = 4.0503$

$$a_1 = \frac{I + II}{V} \qquad b_1 = \frac{III - IV}{V}$$

$$a_o = X_s - a_1 x_s + b_1 y_s$$

$$b_o = Y_s - b_1 x_s - a_1 y_s$$

Accuracy Assessment:

$X = a_0 + a_1 x - b_1 y$	$Y = b_0 + b_1 x + a_1 y$	$V_x = X' - X$	$V_y = Y' - Y$	V_x^2	V_y^2
670309.22	224165.37	4.22	-7.63	17.82	58.28
670755.79	223898.46	-8.21	0.46	67.41	0.21
670744.99	224451.17	3.99	7.17	15.92	51.48
	Σ=	0	0	101.14	109.97

$m_x = m_y = 10.27$ $\qquad$ $m_p = 14.53$

4-Parameter **Transformation** with Weight & Over Determination

Math Model

$$X' = a_o + a_1 x - b_1 y$$

$$Y' = b_o + b_1 x + a_1 y$$

Solution

4-Parameter **Transformation** with Weight & Over Determination

Step 1: Computing weighted averages

$$x_s = \frac{\sum_{i=1}^{n} w_i \cdot x_i}{\sum_{i=1}^{n} w_i}$$

$$y_s = \frac{\sum_{i=1}^{n} w_i \cdot y_i}{\sum_{i=1}^{n} w_i}$$

$$X_s = \frac{\sum_{i=1}^{n} w_i \cdot X_i}{\sum_{i=1}^{n} w_i}$$

$$Y_s = \frac{\sum_{i=1}^{n} w_i \cdot Y_i}{\sum_{i=1}^{n} w_i}$$

4-Parameter **Transformation** with Weight & Over Determination

Step 2: Computing Residuals

$$\overline{x_i} = x_i - x_s$$

$$\overline{y_i} = y_i - y_s$$

$$\overline{X_i} = X_i - X_s$$

$$\overline{Y_i} = Y_i - Y_s$$

4-Parameter **Transformation** with Weight & Over Determination

Step 3 Computing a_1 and b_1

$$I = \sum_{i=1}^{n} w_i \cdot \overline{X}_i \cdot \overline{x}_i$$

$$II = \sum_{i=1}^{n} w_i \cdot \overline{Y}_i \cdot \overline{y}_i$$

$$III = \sum_{i=1}^{n} w_i \cdot \overline{Y}_i \cdot \overline{x}_i$$

$$IV = \sum_{i=1}^{n} w_i \cdot \overline{X}_i \cdot \overline{y}_i$$

$$V = \sum_{i=1}^{n} w_i \, (\overline{x}_i^2 + \overline{y}_i^2)$$

$$a_1 = \frac{I + II}{V}$$

$$b_1 = \frac{III - IV}{V}$$

4-Parameter **Transformation** with Weight & Over Determination

Step 4 **Computing a_o and b_o**

$$a_o = X_s - a_1 x_s + b_1 y_s$$

$$b_o = Y_s - b_1 x_s - a_1 y_s$$

Accuracy Assessment

$$V_{x_i} = X_i - X_i^{'} \qquad V_{y_i} = Y_i - Y_i^{'}$$

$$VI = \sum_{i=1}^{n} w_i \cdot V_{x_i}^2 \qquad VII = \sum_{i=1}^{n} w_i \cdot V_{y_i}^2$$

4-Parameter **Transformation** with Weight & Over Determination

Accuracy Assessment of Point *i*

$$m_{o_i} = m_{x_i} = m_{y_i} = \sqrt{\frac{VI + VII}{(2n-4) \cdot w_i}} \qquad m_{p_i} = m_{0_i} \cdot \sqrt{2}$$

Mean Accuracy

$$m_0 = m_x = m_y = \sqrt{\frac{VI + VII}{(2n-4) \cdot \sum_{i=1}^{n} w_i}} \qquad m_p = m_0 \cdot \sqrt{2}$$

Same Problem with 3 points +W

i	x_i	y_i	X_i	Y_i	w_i
1	100.69	166.39	670305	224173	3
2	34.90	56.07	670764	223898	1
3	171.36	58.87	670741	224444	1
				$\Sigma w =$	5

i	$w_i x_i$	$w_i y_i$	$w_i\ X_i$	$w_i\ Y_i$
1	302.07	499.17	2010915	672519
2	34.90	56.07	670764	223898
3	171.36	58.87	670741	224444
$\Sigma =$	508.33	614.11	3352420	1120861

Same Problem with 3 points +W

$\Sigma =$	508.33	614.11	3352420	1120861
	$X_S = \frac{\Sigma}{\Sigma w}$	$Y_S = \frac{\Sigma}{\Sigma w}$	$x_S = \frac{\Sigma}{\Sigma w}$	$y_S = \frac{\Sigma}{\Sigma w}$
	101.666	122.822	670484	224172.2

i	$\bar{x}_i = x_i - x_s$	$\bar{y}_i = y_i - y_s$	$\overline{X}_i = X_i - X_s$	$\overline{Y}_i = Y_i - Y_s$
1	-0.98	43.57	-179.00	0.80
2	-66.77	-66.75	280.00	-274.20
3	69.69	-63.95	257.00	271.80

Same Problem with 3 points +W

i	$w_i \cdot \bar{x}_i \cdot \bar{X}_i$	$w_i \cdot \bar{y}_i \cdot \bar{Y}_i$	$w_i \cdot \bar{x}_i \cdot \bar{Y}_i$	$w_i \cdot \bar{y}_i \cdot \bar{X}_i$	$w_i(\bar{x}_i^2 + \bar{y}_i^2)$
1	524.11	104.56	-2.34	-23396.02	5697.37
2	-18694.48	18303.40	18307.24	-18690.56	8913.53
3	17911.36	-17382.15	18942.83	-16435.66	8947.11
Σ=	-259.01	1025.81	37247.72	-58522.24	23558.01
	I	II	III	IV	V

Same Problem with 3 points +W

Solution:

$$a_1 = \frac{\sum I + \sum II}{\sum V} = \underline{\mathbf{0.03254935}}$$

$$b_1 = \frac{\sum III - \sum IV}{\sum V} = \underline{\mathbf{4.06528246}}$$

$$a_o = X_s - a_1 x_s + b_1 y_s = \underline{\mathbf{670979.997}}$$

$$b_o = Y_s - b_1 x_s - a_1 y_s = \underline{\mathbf{223754.901}}$$

6-Parameter (Affine) **Transformation** with Over Determination

Math Model

$$X' = a_o + a_1 x + a_2 y$$
$$Y' = b_o + b_1 x + b_2 y$$

Solution

Step 1: Computing averages

$$x_s = \frac{\sum_{i=1}^{n} x_i}{n} \qquad y_s = \frac{\sum_{i=1}^{n} y_i}{n} \qquad X_s = \frac{\sum_{i=1}^{n} X_i}{n} \qquad Y_s = \frac{\sum_{i=1}^{n} Y_i}{n}$$

6-Parameter (Affine) **Transformation** with Over Determination

Step 2: Computing Residuals

$$\overline{x_i} = x_i - x_s \qquad \overline{y_i} = y_i - y_s$$

$$\overline{X_i} = X_i - X_s \qquad \overline{Y_i} = Y_i - Y_s$$

6-Parameter (Affine) **Transformation** with Over Determination

Step 2: Computing Residuals

$$\bar{x}_i = x_i - x_s \qquad \bar{y}_i = y_i - y_s$$

$$\overline{X_i} = X_i - X_s \qquad \overline{Y_i} = Y_i - Y_s$$

6-Parameter (Affine) **Transformation** with Over Determination

Step 3 Computing *a_1* and *b_1*

$$a_1 = \frac{\sum_{i=1}^{n} \bar{y}_i^2 \cdot \sum_{i=1}^{n} (\overline{X}_i \cdot \bar{x}_i) - \sum_{i=1}^{n} (\bar{x}_i \cdot \bar{y}_i) \cdot \sum_{i=1}^{n} (\overline{X}_i \cdot \bar{y}_i)}{\sum_{i=1}^{n} \bar{x}_i^2 \cdot \sum_{i=1}^{n} \bar{y}_i^2 - (\sum_{i=1}^{n} \bar{x}_I \cdot \bar{y}_I)^2}$$

$$a_2 = \frac{\sum_{i=1}^{n} \bar{x}_i^2 \cdot \sum_{i=1}^{n} (\overline{X}_i \cdot \bar{y}_i) - \sum_{i=1}^{n} (\bar{x}_i \cdot \bar{y}_i) \cdot \sum_{i=1}^{n} (\overline{X}_i \cdot \bar{x}_i)}{\sum_{i=1}^{n} \bar{x}_i^2 \cdot \sum_{i=1}^{n} \bar{y}_i^2 - (\sum_{i=1}^{n} \bar{x}_I \cdot \bar{y}_I)^2}$$

6-Parameter (Affine) **Transformation** with Over Determination

Step 3 **Computing *a_1* and *b_1***

$$b_1 = \frac{\sum_{i=1}^{n} \bar{y}_i^2 \cdot \sum_{i=1}^{n} (\bar{Y}_i \cdot \bar{x}_i) - \sum_{i=1}^{n} (\bar{x}_i \cdot \bar{y}_i) \cdot \sum_{i=1}^{n} (\bar{Y}_i \cdot \bar{y}_i)}{\sum_{i=1}^{n} \bar{x}_i^2 \cdot \sum_{i=1}^{n} \bar{y}_i^2 - (\sum_{i=1}^{n} \bar{x}_I \cdot \bar{y}_I)^2}$$

$$b_2 = \frac{\sum_{i=1}^{n} \bar{x}_i^2 \cdot \sum_{i=1}^{n} (\bar{Y}_i \cdot \bar{y}_i) - \sum_{i=1}^{n} (\bar{x}_i \cdot \bar{y}_i) \cdot \sum_{i=1}^{n} (\bar{Y}_i \cdot \bar{x}_i)}{\sum_{i=1}^{n} \bar{x}_i^2 \cdot \sum_{i=1}^{n} \bar{y}_i^2 - (\sum_{i=1}^{n} \bar{x}_I \cdot \bar{y}_I)^2}$$

6-Parameter (Affine) **Transformation** with Over Determination

Step 4 **Computing *a_o* and *b_o***

$$a_o = X_s - a_1 x_s - a_2 y_s$$

$$b_o = Y_s - b_1 x_s - b_2 y_s$$

Accuracy Assessment

$$V_{x_i} = X_i - X_i^{'} \qquad V_{y_i} = Y_i - Y_i^{'}$$

$$m_0 = m_x = m_y = \sqrt{\frac{\sum_{i=1}^{n} V_{x_i}^2 + \sum_{i=1}^{n} V_{y_i}^2}{2n - 6}} \qquad m_p = m_0 \cdot \sqrt{2}$$

6-Parameter (Affine) **Transformation** with Over Determination

Accuracy Assessment of Point *i*

$$m_{o_i} = m_{x_i} = m_{y_i} = \sqrt{\frac{VI + VII}{(2n-4)\cdot w_i}}$$

$$m_{p_i} = m_{0_i} \cdot \sqrt{2}$$

Mean Accuracy

$$m_0 = m_x = m_y = \sqrt{\frac{VI + VII}{(2n-4)\cdot \sum_{i=1}^{n} w_i}}$$

$$m_p = m_0 \cdot \sqrt{2}$$

6-Parameter **Transformation** with Weights & Over Determination

Math Model

$$X' = a_o + a_1 x + a_2 y$$
$$Y' = b_o + b_1 x + b_2 y$$

Solution

6-Parameter **Transformation** with Weights & Over Determination

Step 1: Computing weighted averages

$$x_s = \frac{\sum_{i=1}^{n} w_i \cdot x_i}{\sum_{i=1}^{n} w_i}$$

$$y_s = \frac{\sum_{i=1}^{n} w_i \cdot y_i}{\sum_{i=1}^{n} w_i}$$

$$X_s = \frac{\sum_{i=1}^{n} w_i \cdot X_i}{\sum_{i=1}^{n} w_i}$$

$$Y_s = \frac{\sum_{i=1}^{n} w_i \cdot Y_i}{\sum_{i=1}^{n} w_i}$$

6-Parameter **Transformation** with Weights & Over Determination

Step 2: Computing Residuals

$$\overline{x_i} = x_i - x_s$$

$$\overline{y_i} = y_i - y_s$$

$$\overline{X_i} = X_i - X_s$$

$$\overline{Y_i} = Y_i - Y_s$$

6-Parameter **Transformation** with Weights & Over Determination

Step 3 Computing a_1 and b_1

$$I = \sum_{i=1}^{n} w_i \cdot \overline{X}_i \cdot \overline{x}_i$$

$$II = \sum_{i=1}^{n} w_i \cdot \overline{Y}_i \cdot \overline{y}_i$$

$$III = \sum_{i=1}^{n} w_i \cdot \overline{Y}_i \cdot \overline{x}_i$$

$$IV = \sum_{i=1}^{n} w_i \cdot \overline{X}_i \cdot \overline{y}_i$$

$$V = \sum_{i=1}^{n} w_i \cdot \overline{x}_i^{\,2}$$

$$VI = \sum_{i=1}^{n} w_i \cdot \overline{y}_i^{\,2}$$

6-Parameter **Transformation** with Weights & Over Determination

Step 3 Computing a_1 and b_1

$$VII = \sum_{i=1}^{n} w_i \cdot \overline{y}_i \cdot \overline{x}_i$$

$$IX = V \cdot VI - V \cdot III$$

$$a_1 = \frac{VI \cdot I - VII \cdot IV}{IX}$$

$$a_2 = \frac{V \cdot IV - VII \cdot I}{IX}$$

$$b_1 = \frac{VI \cdot III - VII \cdot II}{IX}$$

$$b_2 = \frac{V \cdot II - VII \cdot III}{IX}$$

$$a_o = X_s - a_1 x_s - a_2 y_s$$

$$b_o = Y_s - b_1 x_s - b_2 y_s$$

6-Parameter **Transformation** with Weights & Over Determination

Accuracy Assessment

$$v_{x_i} = X_i - X_i'$$ $$v_{y_i} = Y_i - Y_i'$$

$$XI = \sum w_i \cdot v_{x_i}^2$$ $$XII = \sum w_i \cdot v_{y_i}^2$$

$$m_0 = m_x = m_y = \sqrt{\frac{XI + XII}{(2n-6)\sum w_i}}$$

$$m_p = m_0 \cdot \sqrt{2}$$

$$Scale_x = \sqrt{a_1^2 + b_1^2}$$

$$Scale_y = \sqrt{a_2^2 + b_2^2}$$

The Rigid Body Transformation.

The math Model of a 3 parameter transformation is:

$$X = a_o + x \cdot \cos\alpha - y \cdot \sin\alpha$$

$$Y = b_o + x \cdot \sin\alpha + y \cdot \cos\alpha$$

Since sin α & cos α are non linear variables, for a Least Squares solution, we have to compute approximate value a'$_o$, b'$_o$, and α'. for the unknown a$_o$, b$_o$, and α.

$$X' = a_o' + x \cdot \cos\alpha' - y \cdot \sin\alpha'$$

$$Y' = b_o' + x \cdot \sin\alpha' + y \cdot \cos\alpha'$$

$$V_x = X - X'$$ $$V_y = Y - Y'$$

The Rigid Body Transformation.

To perform the Least Squares solution we need to compute the following constants:

$$a = \sum_{i=1}^{n} w_i$$

$$b = \sum_{i=1}^{n} (b_o^{'} - Y_i^{'}) \cdot w_i$$

$$c = \sum_{i=1}^{n} (X_i^{'} - a_o^{'}) \cdot w_i$$

$$d = \sum_{i=1}^{n} (b_o^{'} - Y_i^{'})^2 \cdot w_i + \sum_{i=1}^{n} (X_i^{'} - a_o^{'})^2 \cdot w_i$$

$$f = a \cdot d - b^2 - c^2$$

$$g = \sum_{i=1}^{n} (w_i \cdot V_{x_i})$$

$$h = \sum_{i=1}^{n} (w_i \cdot V_{y_i})$$

$$m = \sum_{i=1}^{n} (b_o^{'} - Y_i^{'}) \cdot w_i \cdot V_{x_i} + \sum_{i=1}^{n} (X_i^{'} - a_o^{'}) \cdot w_i \cdot V_{y_i}$$

The Rigid Body Transformation.

Finally, the solution for the unknown is:

$$a_o = a'_o + \frac{1}{a \cdot f}[(f + b^2) \cdot g + b \cdot c \cdot h - a \cdot b \cdot m]$$

$$b_o = b'_o + \frac{1}{a \cdot f}[b \cdot c \cdot g + (f + c^2) \cdot h - a \cdot c \cdot m]$$

$$\alpha = \alpha' + \frac{180}{\pi \cdot f}[-b \cdot g - c \cdot h + a \cdot m]$$

(α and α' are in decimal degrees)

The Rigid Body Transformation.

Approximate Accuracy Assessment

$$v_{x_i} = X_i - X_i'$$

$$v_{y_i} = Y_i - Y_i'$$

$$XI = \sum w_i \cdot v_{x_i}^2$$

$$XII = \sum w_i \cdot v_{y_i}^2$$

$$m_0 = m_x = m_y = \sqrt{\frac{XI + XII}{(2n-3)\sum w_i}}$$

$$m_p = m_0 \cdot \sqrt{2}$$

$$Scale_x = \sqrt{a_1^2 + b_1^2}$$

$$Scale_y = \sqrt{a_2^2 + b_2^2}$$

The Rigid Body Transformation.

Computation Example:

Point	*x*	*y*	*X*	*Y*	W_i
A	1000.00	1000.00	999.70	1000.05	100
B	1000.00	1350.00	999.90	1350.33	100
C	1350.00	1350.00	1350.22	1350.00	1
D	1350.00	1000.00	1349.90	1000.11	1
				$\Sigma\, w =$ a	202

Approximate Values

a_o'=	-1.3144
b_o'=	0.20617
α=	-0.032388932

Point	X'	Y'
A	999.25073	999.64072
B	999.44859	1349.64066
C	1349.44853	1349.44281
D	1349.25068	999.44286

The Rigid Body Transformation.

i	$w_i(b'_o-Y_i')$	$w_i(X_i'-a'_o)$	$w_i(b'_o-Y_i')^2$	$w_i(X_i'-a'_o)^2$
A	-99943.45467	100056.5134	99886941.32	100113058.7
B	-134943.4491	100076.2986	182097344.5	100152655.5
C	-1349.236638	1350.76293	1820439.506	1824560.494
D	-999.236694	1350.565078	998473.9707	1824026.029
Σ =	-237235.3771	202834.14	284803199.3	203914300.7
	b	c	I	II

$\boxed{d = I + II}$

The Rigid Body Transformation.

i	$w_i V_x$	$w_i V_y$	$w_i (b'_o-Y_i')V_x$	$w_i (X_i'-a'_o)V_y$
A	44.92663029	40.92832538	-44901.22638	40951.45536
B	45.14135865	68.93391762	-60915.30633	68986.51327
C	0.771469509	0.557191893	-1040.894927	752.6341537
D	0.649322225	0.66713597	-648.8265938	901.0105435
Σ =	91.48878067	111.0865709	-107506.2542	111591.6133
	g	h	III	IV

$\boxed{m = III + IV}$

The Rigid Body Transformation.

Computation Constants

a =	202
b =	-237235.3771
c =	202834.14
d =I+II=	488717500
f =	1298622500
g =	91.48878068
h =	111.0865709
m =III+IV=	4085.359084

Final Transformation Parameters

a_o =	-0.863774335
b_o =	0.758060671
α =	-0.000111675

Application of coordinate transformation

Methods for Transforming NAD27 to NAD83

- NADCON
  - Accuracy of coordinate transformation using NADCON is often not better than 1 foot (0.3 meters).
- Global, Regressions, 3D Transformations
- 6 parameter transformation that can yield a better local fit and consistency in coordinate values.

Application of coordinate transformation

6 parameter transformation for NAD coordinates:

- The method assumes that the points are on a plane, thus, control points that do not extend beyond a radius of, say, 5 miles. Beyond this radius a non-linear model should be considered.
- The state plane coordinates of NAD27 were defined in terms of X and Y. In NAD83 they are defined in terms of N (northing) and E (easting). The X coordinate corresponds to E and the Y coordinate corresponds to N. Thus, when using our transformation method NAD83 coordinates should be ordered as E,N and not N,E.

Application of coordinate transformation

6 parameter transformation for NAD coordinates:

- Another application of this transformation method with respect to SPCS is a transformation of an entire project from sate plane coordinates to ground values and visa versa. The relationship between ground values and state plane values is a common problem in construction surveys and GIS.

NGS Data Sheet

KV6828 ***
KV6828 DESIGNATION - 7 A 1
KV6828 PID - KV6828
KV6828 STATE/COUNTY- NJ/ESSEX
KV6828 USGS QUAD - ORANGE (1982)
KV6828
KV6828 HORZ DATUM - NAD 83
KV6828
KV6828 POSITION - 40 48 31.85755(N) 074 09 24.19389(W) ADJUSTED
KV6828 83 minus 27 - +00.36106 -01.48170 NADCON
KV6828

		North	East	Scale	Converg.
KV6828;SPC NJ	-	219,378.713	178,962.290	0.99991032	+0 13 27.7

NGS Data Sheet

	North	East	Scale	Converg.
KV6831;SPC NJ	214,714.841	172,089.086	0.99990600	+0 10 15.1
KV6840;SPC NJ	211,463.099	175,785.386	0.99990818	+0 11 57.3
KV6841;SPC NJ	214,288.325	176,787.409	0.99990883	+0 12 25.8
KV6846;SPC NJ	215,614.614	174,999.825	0.99990769	+0 11 36.3
KV6849;SPC NJ	213,463.009	174,509.622	0.99990739	+0 11 22.2
KV6850;SPC NJ	213,385.955	174,810.878	0.99990757	+0 11 30.6
KV6857;SPC NJ	212,161.942	174,184.552	0.99990720	+0 11 12.9
KV6858;SPC NJ	212,845.678	174,278.862	0.99990725	+0 11 15.6

WEIGHTED TRANSFORMATION from NAD27 to NAD83 (6 parameters)

	x	y	X (E)	Y (N)	w
NEWARK 1	2125650.302	696339.488	174184.552	212161.942	1
EAST ORANGE 2	2127697.804	700359.394	174810.878	213385.955	6
CLINTON	2128303.873	707672.831	174999.825	215614.614	5
NEWARK 89 1	2130907.473	694056.613	175785.386	211463.099	2
NEWARK 89196	2134177.740	703332.456	176787.409	214288.325	3
Σ	10646737.192	3501760.782	876568.050	1066913.935	17

WEIGHTED TRANSFORMATION from NAD27 to NAD83 (6 parameters)

	$w \cdot x$	$w \cdot y$	$w \cdot X$	$w \cdot Y$
NEWARK 1	2125650.302	696339.488	174184.552	212161.942
EAST ORANGE 2	12766186.824	4202156.364	1048865.268	1280315.730
CLINTON	10641519.365	3538364.155	874999.125	1078073.070
NEWARK 89 1	4261814.946	1388113.226	351570.772	422926.198
NEWARK 89196	6402533.220	2109997.368	530362.227	642864.975
Σ	36197704.657	11934970.601	2979981.944	3636341.915

x_S	y_S	X_S	Y_S
2129276.745	702057.094	175293.056	213902.466

WEIGHTED TRANSFORMATION from NAD27 to NAD83 (6 parameters)

	x_i	y_i	X_i	Y_i
NEWARK 1	-3626.443	-5717.606	-1108.504	-1740.524
EAST ORANGE 2	-1578.941	-1697.700	-482.178	-516.511
CLINTON	-972.872	5615.737	-293.231	1712.148
NEWARK 89 1	1630.728	-8000.481	492.330	-2439.367
NEWARK 89196	4900.995	1275.362	1494.353	385.859
Σ	353.469	-8524.689	102.772	-2598.393

WEIGHTED TRANSFORMATION from NAD27 to NAD83 (6 parameters)

	$w_i x_i X_i$ (I)	$w_i y_i Y_i$ (II)	$w_i x_i Y_i$ (III)	$w_i y_i X_i$ (IV)
NEWARK 1	4019924.343	9951628.418	6311908.764	6337986.626
EAST ORANGE 2	4567977.861	5261280.701	4893237.010	4911557.261
CLINTON	1426378.168	48074874.416	-8328502.220	-8233527.409
NEWARK 89 1	1605714.631	39032212.943	-7955889.091	-7877761.325
NEWARK 89196	21971458.772	1476331.089	5673285.688	5717524.102
Σ	33591453.776	103796327.568	594040.151	855779.255

WEIGHTED TRANSFORMATION from NAD27 to NAD83 (6 parameters)

	$w_i\, x_i^2$ (V)	$w_i\, y_i^2$ (VI)	$w_i\, x_i y_i$ (VII)
NEWARK 1	13151085.419	32691020.389	20734570.205
E. ORANGE 2	14958319.173	17293115.335	16083405.693
CLINTON	4732395.064	157682500.356	-27316952.361
NEWARK 89 1	5318550.690	128015398.110	-26093224.866
NEWARK 89196	72059269.808	4879643.343	18751627.561
Σ	110219620.153	340561677.533	2159426.232

$IX = V \cdot VI - VII^2$ =4973688528744000.000

6-Parameter **Transformation** with Weights & Over Determination

Step 3 Computing a_1 and b_1

$$\boxed{VII = \sum_{i=1}^{n} w_i \cdot \overline{y}_i \cdot \overline{x}_i}$$

$$\boxed{IX = V \cdot VI - VII^2}$$

WEIGHTED TRANSFORMATION from NAD27 to NAD83 (6 parameters)

The Transformation parameters are:

$a_0 =$	-474026.438	$b_0 =$	1165.690
$a_1 =$	0.3047570	$b_1 =$	-0.0005817
$a_2 =$	0.0005804	$b_2 =$	0.3047835

$$a_1 = \frac{VI \cdot I - VII \cdot IV}{IX} \qquad a_2 = \frac{V \cdot IV - VII \cdot I}{IX}$$

$$b_1 = \frac{VI \cdot III - VII \cdot II}{IX} \qquad b_2 = \frac{V \cdot II - VII \cdot III}{IX}$$

$$a_o = X_s - a_1 x_s - a_2 y_s \qquad b_o = Y_s - b_1 x_s - b_2 y_s$$

WEIGHTED TRANSFORMATION from NAD27 to NAD83 (6 parameters)

Coordinates from Parameters

$X=a_0+a_1x+a_2y$	$Y=b_0+b_1x+b_2y$
174184.553	212161.943
174810.877	213385.953
174999.826	215614.616
175785.388	211463.102
176787.408	214288.324

Checking Closure:

	$V_x=X_{cmp}-X_{giv}$	$V_y=Y_{cmp}-Y_{giv}$
	0.001	0.001
	-0.001	-0.002
	0.001	0.002
	0.002	0.003
	-0.001	-0.001
$\Sigma=$	0.000	0.000
$\Sigma^2=$	0.000004	0.000015

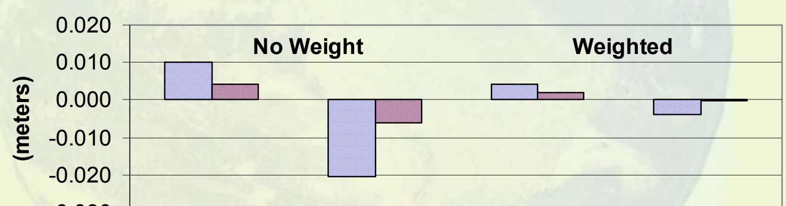
WEIGHTED TRANSFORMATION from NAD27 to NAD83 (6 parameters)

Transformation Check with an Independent Point

		X (E)	Y (N)
NEWARK 2	Computed	174278.860	212845.676
	Data Sheet	174278.862	212845.678
	Δ	-0.002	-0.002
EAST ORANGE 1	Computed	174509.621	213463.007
	Data Sheet	174509.622	213463.009
	Δ	-0.001	-0.002

WEIGHTED TRANSFORMATION from NAD27 to NAD83 (6 parameters)

Differences between computed and given coordinate values



	East Orange 1	Newark 2	East Orange 1	Newark 2
dE	0.010	-0.020	0.004	-0.004
dN	0.004	-0.006	0.002	0.000

Analysis of multiple evidence

When a professional land surveyor is engaged in determining the boundaries of a specific parcel he/she has the duty to gather as much information as possible on the parcel in question and on the surrounding parcels. Information on boundaries can be found in deeds, subdivision maps, court records and so on. After sorting the evidence, the professional land surveyor proceeds by trying to recover as many parcel corners and physical evidence as possible in the field and perform a survey to map these evidences. The final boundary is determined based on weighing the evidence according to their relative importance.

Analysis of multiple evidence

A problem with recovering a large amount of evidences is how to analyze them in a sole, comprehensive fashion. Some surveyors avoid this kind of data acquisition because of their inability to perform this type of analysis. Others, use a trial and error approach for sorting evidence. The trial and error approach is based on holding the location of 2 points fixed and trying to fit all other points of the survey to the existing record data. This process is repeated for several combinations of fixed points until a "best fit" is found. The best fit is then used for determining the boundary.

Analysis of multiple evidence

The concept of coordinate transformation presented here provides a comprehensive treatment of evidence. This approach will be described next.

Using Coordinate Transformation for Boundary Determination.

Research for Evidence (Office)

Search for Evidence (Field)

Field Survey of Evidence and Computations

COGO and Plotting of Research Evidence

Evidence Transformation: COGO → Survey

Using Coordinate Transformation for Boundary Determination.

ANALYSIS

Corner Transformation COGO → Survey

Inverse Computations

Corner Stake-out

Using Coordinate Transformation for Boundary Determination.

Step 1: Survey found evidence and compute coordinates for these points in a local coordinate system.

Step 2: Evaluate the relative importance of the evidence and assign weights for each one of them.

Step 3: Compute filed map coordinates (using COGO or another method) for the evidence and for the parcel corners.

Using Coordinate Transformation for Boundary Determination.

Step 4: Transform map coordinates of the evidence to survey coordinates and analyze the residuals. If there are large residuals at some points, repeat the transformation computation without the “suspect” points.

Step 5: After completing and accepting the transformation results, use the computed transformation parameters to transform the corners of the parcel (or parcels) to the survey coordinate system.

Using Coordinate Transformation for Boundary Determination.

Step 6: Compute inverses between the parcel coordinates and survey stations for stake out notes.

Step 7: Mark parcel corners.

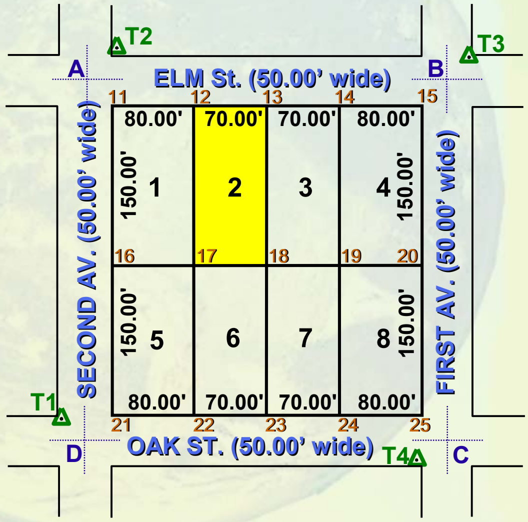
Application in boundary surveying, Case I, No monuments found

Traverse Coordinates

Pt.	X	Y
T1	1820.93	2321.31
T2	1531.92	2533.02
T3	1680.55	2849.68
T4	2038.96	2627.84

Sideshots Coordinates

Pt.	X	Y
A	1544.50	2499.60
B	1692.12	2816.69
C	2009.25	2669.08
D	1861.64	2351.95



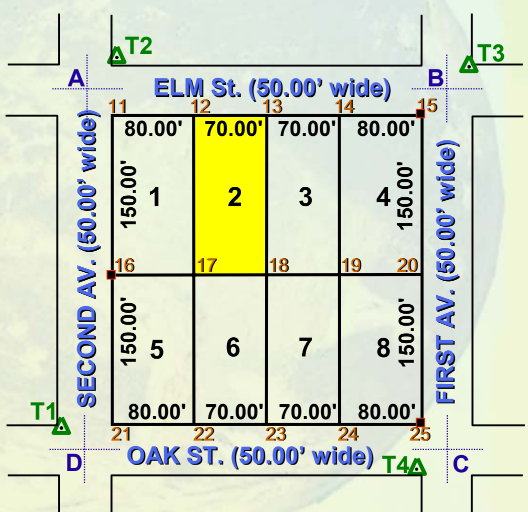
Application in boundary surveying, Case II, 3 monuments found

Traverse Coordinates

Pt.	X	Y
T1	1820.93	2321.31
T2	1531.92	2533.02
T3	1680.55	2849.68
T4	2038.96	2627.84

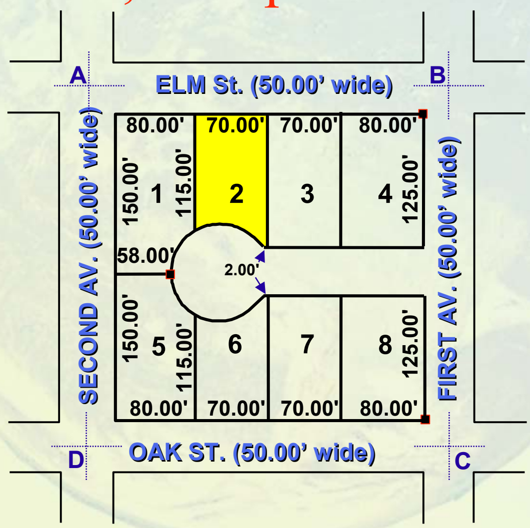
Sideshots Coordinates

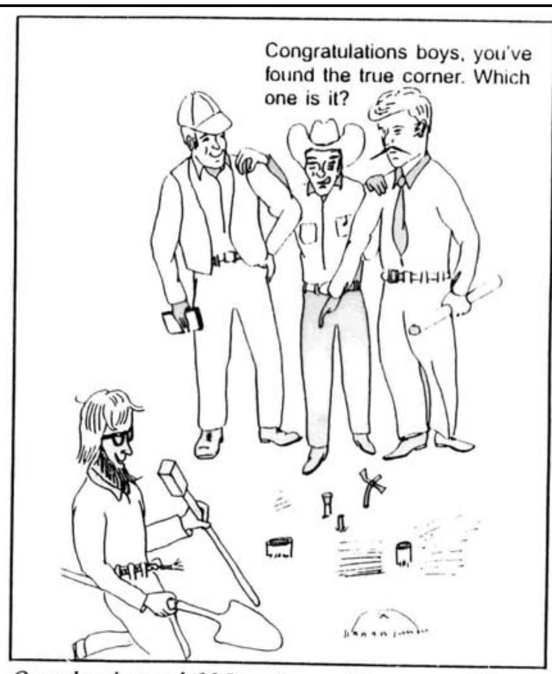
Pt.	X	Y
15	1704.25	2783.60
16	1713.70	2448.40
25	1976.03	2656.93



Application in boundary surveying, Case III, Complex Geometry

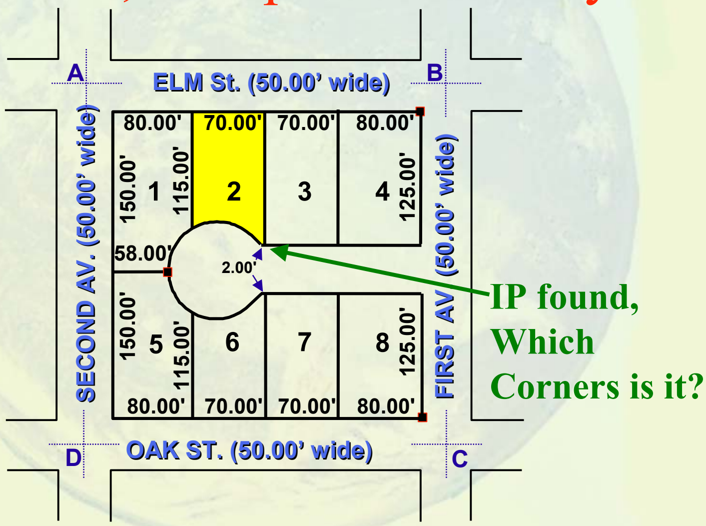
Define Corners of Lot 2?





Our thanks and $25 to Larry Thurman of Las

Application in boundary surveying, Case IV, Complex Geometry



Using Coordinate Transformation for Boundary Determination.

3 MONUMENT CASE (similarity 4 parameters)

	x	y	X	Y
15	1325.00	1325.00	1704.25	2783.60
25	1325.00	1025.00	1976.03	2656.93
16	1025.00	1175.00	1713.70	2448.40
Σ=	3675.00	3525.00	5393.98	7888.93

x_s	y_s	X_s	Y_s
1225.00	1175.00	1797.99	2629.64

Using Coordinate Transformation for Boundary Determination.

3 MONUMENT CASE (similarity 4 parameters)

	x_i	y_i	X_i	Y_i
	100.00	150.00	-93.74	153.96
	100.00	-150.00	178.04	27.29
	-200.00	0.00	-84.29	-181.24
Σ=	0.00	0.00	0.00	0.00

	x_iX_i	y_iY_i	x_iY_i	y_iX_i	$x_i^2+y_i^2$
	-9374.33	23093.50	15395.67	-14061.50	32500.00
	17803.67	-4093.00	2728.67	-26705.50	32500.00
	16858.67	0.00	36248.67	0.00	40000.00
Σ=	25288.00	19000.50	54373.00	-40767.00	105000.00

Using Coordinate Transformation for Boundary Determination.

3 MONUMENT CASE (similarity 4 parameters)

$a_0 =$ 2345.956 $\quad$ $b_0 =$ 1024.067

$a_1 =$ 0.421795 $\quad$ $b_1 =$ 0.906095

	X_{COMP}	Y_{COMP}	V_x	V_x^2	V_y	V_y^2
15	1704.26	2783.52	0.01	0.0001	-0.08	0.0061
25	1976.09	2656.98	0.06	0.0033	0.05	0.0029
16	1713.63	2448.42	-0.07	0.0043	0.02	0.0006
		Σ=	0.00	0.0077	0.00	0.0095

SDxy=0.09 **SDp=0.13**

Using Coordinate Transformation for Boundary Determination.

Computing "Survey" coordinates for Lot corners

	X'	Y'
11	1577.72	2511.69
12	1611.46	2584.18
13	1640.99	2647.61
14	1670.51	2711.03
15	1704.26	2783.52
16	1713.63	2448.42
17	1747.38	2520.91
18	1776.90	2584.34

	X'	Y'
T1	1820.93	2321.31
T2	1531.92	2533.02
T3	1680.55	2849.68
T4	2038.96	2627.84
A	1544.52	2499.59
B	1692.15	2816.72
C	2009.28	2669.09
D	1861.66	2351.96

Using Coordinate Transformation for Boundary Determination.

Computing "Survey" coordinates for Lot 2

	X'	Y'
12	1611.46	2584.18
13	1640.99	2647.61
17	1747.38	2520.91
18	1776.90	2584.34

12-13	69.96
13-18	149.92
18-17	69.96
17-12	149.92

Using Coordinate Transformation for Boundary Determination.

Computing "Survey" coordinates for Case III

Original			New (survey)	
18a	1173.00	1175.00	1776.06	2582.53
17a	1105.00	1210.00	1715.66	2535.67